

Analytical results for group averaged scattering cross sections of high temperature plasmas

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Multi-group method is an accepted technique for approximately solving the equation of radiative transfer. In this paper, group averaged transfer scattering cross sections, required for solving the equation of radiative transfer in a multi-group approach, are presented. Compton scattering is approximately described by the Compton cross section for free electrons at rest. In the low photon energy limit analytical results for scattering coefficients have been derived.

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I. INTRODUCTION

Thermal radiation pressure, energy density and flux are determined by the specific intensity which is the solution of the equation of radiative transfer (ERT) [1]. Solving the full equation of radiative transfer is a complex and cost intensive task. Several methods of solving the ERT approximately have been studied in literature. In regions where the mean free path of photons is small compared to the dimension of the system, diffusion techniques are used widely. These methods have been simplify further by taking overall frequency spectra integrated quantities into account [2]. To effectively reduce technical effort, diffusion methods combined with flux limiters have been applied to optical thin systems too. Morel [3] showed that the accuracy of such methods is of zeroth and first order compared to the exact transfer asymptotics. To enhance accuracy while solving the ERT one applies the multi-group method. This method divides the frequency spectrum into several frequency groups. All photons belonging to one frequency group have the same frequency-independent averaged properties that are characteristic for that group. This method is at least accurate for coarse frequency groups. The purpose of this paper is to derive analytically multi-group scattering and transfer scattering cross section in low photon energy limit. In the first part the derivation of the multi-group ERT is repeated. Based on that results various analytical opacity functions for the scattering contributions are found. It is shown, that all appearing expressions for the scattering opacities can be reduced to the problem of solving a special integral type. The results are compared with early six-group results obtained numerically by Pritzker. The extension of the achieved opacity functions to fine structured frequency groups is straightforward.

II. MULTI-GROUP METHOD

A. Group integrated quantities

The radiation transport equation neglecting induced scattering terms reads [1]

$$\frac{1}{c} \frac{\partial I_\nu(\mathbf{\Omega})}{\partial t} + \mathbf{\Omega} \cdot \nabla I_\nu(\mathbf{\Omega}) = \Sigma'_a(\nu) (B_\nu(\theta) - I_\nu(\mathbf{\Omega})) - \Sigma_s(\nu) I_\nu(\mathbf{\Omega}) + \int_0^\infty d\nu' \int_{4\pi} d\mathbf{\Omega}' \frac{\nu}{\nu'} \Sigma_s(\nu' \rightarrow \nu, \mathbf{\Omega}' \cdot \mathbf{\Omega}) I_{\nu'}(\mathbf{\Omega}'). \quad (1)$$

$I_\nu \equiv I_\nu(\mathbf{r}, \mathbf{\Omega}, t)$ is the specific intensity, c the speed of light, t the time, ν the photon frequency, Σ'_a the macroscopic absorption coefficient corrected for induced emission effects, $\theta = k_B T$ the kinetic temperature, k_B the Boltzmann constant, T the temperature, $\mathbf{\Omega}$ the direction of flight of the photons and Σ_s the macroscopic scattering coefficient. The source function is of Planckian type and given by $B_\nu(\theta)$

$$B_\nu(\theta) = \frac{2h\nu^3}{c^2} (\exp(h\nu/\theta) - 1)^{-1}. \quad (2)$$

h is the Planck constant. The spatial and time dependencies in I_ν are suppressed. Following Pritzker et al. [4] the photon spectrum is divided into six energy groups. The energy groups are presented in table (I). The group identifier is g .

g	1	2	3	4	5	6
$h\nu_g$	5.00e+02	6.40e+01	1.60e+01	4.00e+00	1.00e+00	2.50e-01
$h\nu_{g+1}$	6.40e+01	1.60e+01	4.00e+00	1.00e+00	2.50e-01	1.00e-04

TABLE I. Photon energy groups. The photon energy is given in keV. $h\nu_g$ means the upper photon energy boundary and $h\nu_{g+1}$ means the lower photon energy boundary.

The integration of (1) over the energy groups leads to

$$\left(\frac{1}{c} \frac{\partial}{\partial t} + \mathbf{\Omega} \cdot \nabla \right) I_g(\mathbf{\Omega}) = - (\Sigma'_{ag} + \Sigma_{stg}) I_g(\mathbf{\Omega}) + Q_g + \int_g d\nu \mathcal{C} \quad (3)$$

where $\int_g d\nu [\dots]$ is a shortened term for $\int_{\nu_{g+1}}^{\nu_g} d\nu [\dots]$. \mathcal{C} indicates the in-scattering contribution

$$C = \int_0^\infty d\nu' \int_{4\pi} d\Omega' \frac{\nu}{\nu'} \Sigma_s(\nu' \rightarrow \nu, \Omega' \cdot \Omega) I_{\nu'}(\Omega') \quad (4)$$

and will be considered later. The variables Σ'_{ag} , I_g , B_g , Q_g , Σ_{stg} denote the frequency group (see table (I)) integrated quantities of the macroscopic absorption coefficient, specific intensity, black body radiation, emission source and the macroscopic scattering coefficient, respectively. Formally, those quantities are given by [1]

$$I_g(\Omega) = \int_g d\nu I_\nu(\Omega) \quad (5)$$

$$\Sigma'_{ag} = \left(\int_g d\nu \Sigma'_a(B_\nu(\theta) - I_\nu) \right) \times \left(\int_g d\nu (B_\nu(\theta) - I_\nu) \right)^{-1} \quad (6)$$

$$\Sigma_{stg} = \frac{1}{I_g} \int_g d\nu \Sigma_s(\nu) I_\nu \quad (7)$$

$$Q_g = \int_g d\nu \Sigma'_a(\nu) B_\nu(\theta) \quad (8)$$

$$B_g(\theta) = \int_g d\nu B_\nu(\theta). \quad (9)$$

In a local thermal equilibrium regime Σ'_{ag} and Σ_{stg} can be approximated by their Planck averaged quantities. In this case Σ'_{ag} and Σ_{stg} read

$$\Sigma'_{ag} = \frac{1}{B_g} \int_g d\nu \Sigma'_a(\nu) B_\nu(\theta) \quad (10)$$

$$\Sigma_{stg} = \frac{1}{B_g} \int_g d\nu \Sigma_s(\nu) B_\nu(\theta). \quad (11)$$

Generally, the radiation source is given by the ratio of the emission and absorption coefficient. In local thermal equilibrium the source is of Planckian type. Therefore one has for the group emission source

$$Q_g = \int_g d\nu \Sigma'_a(\nu) B_\nu(\theta) = \Sigma'_{ag} B_g. \quad (12)$$

The coefficients Σ'_{ag} and Q_g are not considered within this report. These contributions are discussed widely in [4] and [5].

B. Group integrated Planck function

Inserting the Planck function in (9) leads to

$$B_g = \frac{2\theta^4}{c^2 h^3} \int_{u_{g+1}}^{u_g} du u^3 (\exp(u) - 1)^{-1} \quad (13)$$

where u_g and u are defined as $u_g = h\nu_g/\theta$ and $u = h\nu/\theta$. In contrast to the numerical evaluation of (9) in [4] an analytical solution of this integral is possible. A discussion of this integral can be found in the appendix. Solving the integral yields

$$B_g = \frac{2\theta^4}{c^2 h^3} \mathcal{I}_3(u_{g+1}, u_g). \quad (14)$$

The results for the six group Planck spectrum B_g in units of keV/cm²s are presented in table (II).

g/θ	1.00e-01	3.16e-01	1.00e+00	3.16e+00	1.00e+01
1	9.329e-243	2.911e-52	1.387e+09	4.852e+28	2.246e+35
2	4.277e-36	4.428e+12	1.758e+28	4.834e+33	1.593e+36
3	9.228e+14	2.585e+27	8.230e+31	1.426e+34	2.194e+35
4	1.951e+26	1.161e+30	1.149e+32	1.251e+33	5.657e+33
5	1.443e+28	8.352e+29	6.923e+30	2.887e+31	9.936e+31
6	5.801e+27	3.802e+28	1.490e+29	5.025e+29	1.623e+30

TABLE II. Six-group Planck spectrum B_g in units of keV/cm²s. g is the group index and θ is the temperature keV. No contributions to B_g are given for high energy groups and small temperatures. In that case Pritzker et al. [4] used a transformed presentation for $B_\nu(\theta)$ called $B_\nu^*(\theta)$. To avoid numerical problems while evaluating group constants B_g^* is defined by $B_\nu^*(\theta) = \exp(u_{g+1})B_\nu(\theta)$. In the present analytical considerations no such transformations are necessary.

C. Scattering cross section

The scattering contribution appearing in (3) is obtained by using the angle integrated Compton cross section [1]

$$\Sigma_s(\nu) = \frac{3}{4}\Sigma_{Th} \left\{ \left(\frac{1+\gamma}{\gamma^3} \right) \left[\frac{2\gamma(1+\gamma)}{1+2\gamma} - \ln(1+2\gamma) \right] + \frac{1}{2\gamma} \ln(1+2\gamma) - \frac{1+3\gamma}{(1+2\gamma)^2} \right\} \quad (15)$$

where $\Sigma_{Th} = 0.665 \times 10^{-24} n_e$ is the macroscopic Thomson cross section, n_e the electron density and $\gamma = h\nu/m_e c^2$. Using (11) and the abbreviation $\gamma_g = h\nu_g/m_e c^2$ one obtains the Planck averaged scattering contribution

$$\begin{aligned} \Sigma_{stg} = \frac{\Sigma_{Th}}{B_g} \left(\frac{3(m_e c^2)^4}{2c^2 h^3} \right) \int_{\gamma_{g+1}}^{\gamma_g} \frac{d\gamma \gamma^3}{(\exp(\gamma m_e c^2/\theta) - 1)} \left\{ \left(\frac{1+\gamma}{\gamma^3} \right) \times \right. \\ \left. \times \left[\frac{2\gamma(1+\gamma)}{1+2\gamma} - \ln(1+2\gamma) \right] + \frac{1}{2\gamma} \ln(1+2\gamma) - \frac{1+3\gamma}{(1+2\gamma)^2} \right\}. \end{aligned} \quad (16)$$

The integration is carried out numerically. The results are given in table (IV). In case of small photon energies, e.g. $\gamma \ll 1$, equation (16) is integrated analytically. The scattering cross section (15) corrected to second order is [1]

$$\Sigma_s = \Sigma_{Th} \left(1 - 2\gamma + \frac{26}{5}\gamma^2 \right). \quad (17)$$

Inserting the above expression in (11) yields

$$\begin{aligned} \Sigma_{stg} = \frac{\Sigma_{Th}}{B_g} \left(\frac{2(m_e c^2)^4}{c^2 h^3} \right) \left\{ \int_{\gamma_{g+1}}^{\gamma_g} \frac{d\gamma \gamma^3}{(\exp(\gamma m_e c^2/\theta) - 1)} \right. \\ \left. - 2 \int_{\gamma_{g+1}}^{\gamma_g} \frac{d\gamma \gamma^4}{(\exp(\gamma m_e c^2/\theta) - 1)} + \frac{26}{5} \int_{\gamma_{g+1}}^{\gamma_g} \frac{d\gamma \gamma^5}{(\exp(\gamma m_e c^2/\theta) - 1)} \right\}. \end{aligned} \quad (18)$$

Σ_{stg} is related to

$$\Sigma_{stg} = \frac{\Sigma_{Th}}{B_g} \left(\frac{2\theta^4}{c^2 h^3} \right) \left\{ \mathcal{I}_3(u_{g+1}, u_g) - 2\alpha \mathcal{I}_4(u_{g+1}, u_g) + \frac{26}{5} \alpha^2 \mathcal{I}_5(u_{g+1}, u_g) \right\} \quad (19)$$

where α and u_g are given by $\alpha = \theta/m_e c^2$ and $u_g = \gamma_g/\alpha$. The results of expression (19) are presented in table (V).

D. Scattering transfer cross section

The scattering part \mathcal{C} of (1) reads

$$\mathcal{C}[I_\nu] = \int_{4\pi} d\Omega' \int_0^\infty d\nu' \frac{\nu}{\nu'} \Sigma_s(\nu' \rightarrow \nu, \Omega \cdot \Omega') I_{\nu'}(\Omega'). \quad (20)$$

The integration of $\mathcal{C}[I_\nu]$ over ν by splitting the integrals in different photon energy intervals gives the frequency averaged in-scattering contribution

$$\mathcal{C}_g[I_\nu] = \int_{4\pi} d\Omega' \sum_{g'=gmin(g)}^{gmax(g)} S_{g'g} \quad (21)$$

where $S_{g'g}$ is defined by

$$S_{g'g} = \frac{1}{I_{g'}} \int_{g'} d\nu' \int_g d\nu \frac{\nu}{\nu'} \Sigma_s(\nu' \rightarrow \nu, \Omega' \cdot \Omega) I_{\nu'}(\Omega'). \quad (22)$$

$gmin(g)$ indicates the smallest and $gmax(g)$ the highest energy group g' from which a photon is able to scatter into group g . By the summation over g' all possible photon scattering contributions from the energy group g' into the energy group g are considered. $S_{g'g}$ is the group integrated differential scattering coefficient. In local thermal equilibrium the weighting function $I_{\nu'}$ is given by Planck's function approximately. One therefore obtains for $S_{g'g}$

$$S_{g'g} = \frac{1}{B_{g'}} \int_{g'} d\nu' \frac{B_{\nu'}}{\nu'} \int_g d\nu \nu \Sigma_s(\nu' \rightarrow \nu, \Omega' \cdot \Omega). \quad (23)$$

The Legendre expansion of $\Sigma_s(\nu' \rightarrow \nu, \Omega' \cdot \Omega)$ reads

$$\Sigma_s(\nu' \rightarrow \nu, \Omega' \cdot \Omega) = \sum_{l=0}^{\infty} \frac{2l+1}{4\pi} \Sigma_{sl}(\nu' \rightarrow \nu) P_l(\Omega' \cdot \Omega). \quad (24)$$

where the expansion coefficients $\Sigma_{sl}(\nu' \rightarrow \nu)$ are defined by

$$\Sigma_{sl}(\nu' \rightarrow \nu) = 2\pi \int_{-1}^1 d\mu_0 \Sigma_s(\nu' \rightarrow \nu, \mu_0) P_l(\mu_0) \quad (25)$$

where $\mu_0 = \Omega' \cdot \Omega$ is the scattering angle. By Using (24) and (25) in (23) one defines the Legendre ordered group integrated differential scattering coefficient

$$S_{lg'g} = \frac{2\pi}{B_{g'}} \int_1^{-1} d\mu_0 P_l(\mu_0) \int_{g'} d\nu' \frac{B_{\nu'}}{\nu'} \int_g d\nu \nu \Sigma_s(\nu' \rightarrow \nu, \mu_0). \quad (26)$$

l marks the Legendre order. By using the above transformations the differential scattering contribution of the radiative transfer equation reads

$$\mathcal{C}_g[I_\nu] = \sum_{l=0}^{\infty} \frac{2l+1}{4\pi} \int_{4\pi} d\Omega' P_l(\Omega' \cdot \Omega) \sum_{g'=gmin(g)}^{gmax(g)} S_{lg'g} I_{g'}(\Omega'). \quad (27)$$

In the next section the contributions $S_{lg'g}$ are evaluated. Beside numerical calculations analytical expressions are derived.

E. Moments of the group transfer scattering cross section

The differential scattering moments are given by (26). For further purposes $S_{lg'g}$ is transformed by help of (2) and $\gamma = h\nu/m_e c^2$ in

$$S_{lg'g} = \frac{4\pi(m_e c^2)^3}{h^2 c^2} \frac{1}{B_{g'}} \int_{-1}^1 d\mu_0 P_l(\mu_0) \int_{g'} \frac{d\gamma' \gamma'^2}{(\exp(\gamma' m_e c^2 / \theta) - 1)} \times \int_g d\nu \nu \Sigma_s(\nu' \rightarrow \nu, \mu_0). \quad (28)$$

1. *Group transfer scattering cross section in the Thomson limit*

The Thomson differential cross section is given by [1]

$$\Sigma_s(\nu' \rightarrow \nu, \mu_0) = \frac{3}{16\pi} \Sigma_{Th}(1 + \mu_0^2) \delta(\nu' - \nu). \quad (29)$$

Using the above expression in (28) yields

$$S_{lg'g} = \frac{2\pi}{B_{g'}} \int_{-1}^1 d\mu_0 P_l(\mu_0) \int_{g'} d\nu' \frac{B_{\nu'}}{\nu'} \int_g d\nu \nu \frac{3}{16\pi} \Sigma_{Th}(1 + \mu_0^2) \delta(\nu' - \nu) = \frac{3}{8} \Sigma_{Th} \int_{-1}^1 d\mu_0 P_l(\mu_0)(1 + \mu_0^2).$$

The Thomson cross section does not depend on frequency of the incident photons. No frequency shifts occur during the scattering. Hence, the out-scattered photons belong to the same energy group as the in-scattered photons. Owing to the definition of the Legendre polynomials one calculates

$$1 + \mu_0^2 = \frac{4}{3} P_0(\mu_0) + \frac{2}{3} P_2(\mu_0) \quad (30)$$

and therefore

$$\begin{aligned} S_{0gg} &= \frac{\Sigma_{Th}}{2} \int_{-1}^1 d\mu_0 P_0(\mu_0) P_0(\mu_0) = \Sigma_{Th} \\ S_{1gg} &= 0 \\ S_{2gg} &= \frac{\Sigma_{Th}}{4} \int_{-1}^1 d\mu_0 P_2(\mu_0) P_2(\mu_0) = \frac{1}{10} \Sigma_{Th} \end{aligned} \quad (31)$$

or in terms of the Thomson unit $S_{0gg} = 1$, $S_{1gg} = 0$, $S_{2gg} = 1/10$. The orthogonality relation

$$\int_{-1}^1 d\mu_0 P_n(\mu_0) P_m(\mu_0) = \frac{2}{2n+1} \delta_{nm}$$

has been used while evaluating the transfer cross section. All higher moments are zero.

2. *Frequency shift formula*

The shift of frequency of the photon scattering on an electron at rest is given by

$$\Delta\lambda = \lambda - \lambda' = \frac{h}{m_e c} (1 - \mu_0). \quad (32)$$

In that case λ and λ' denote the wavelength of the scattered and in-scattered photon, respectively ($\lambda \geq \lambda'$, $\Delta\lambda \geq 0$). Switching to the energy group notation, using the relation $\lambda = c/\nu$ and considering extremal cases only, one obtains

$$\frac{1}{\nu_g} - \frac{1}{\nu'_{g,max}} = \frac{2h}{m_e c^2} \quad \mu_0 = -1 \quad (33)$$

$$\frac{1}{\nu_g} - \frac{1}{\nu'_{g,min}} = 0 \quad \mu_0 = 1. \quad (34)$$

Therefore one gains

$$\nu'_{max,g} = \frac{\nu_g}{1 - 2\gamma_g} \quad (35)$$

$$\nu'_{min,g} = \nu_g, \quad (36)$$

where γ_g is defined by $\gamma_g = h\nu_g/m_e c^2$. There is no up-scattering for electrons at rest, hence $S_{lg'g} = 0$, if $g' > g$. Table (III) shows the highest frequencies $\nu'_{max,g}$ of photons which are scattered into group g . Comparing the values for $h\nu'_{max,g}$ and $h\nu_{g-1}$ one recognises $h\nu'_{max,g} < h\nu_{g-1}$. This depends on the chosen energy groups. Therefore one has in-scattering contributions from group $g-1$ only. $S_{lg'g} = 0$, if $g' < g-1$. Because of no up-scattering we only have to calculate S_{lg-1g} and S_{lgg} . All other contributions are zero. Pritzker et al. [4] take ν_{g+1} for $\nu'_{min,g}$. This is inappropriate. Their usage leads to a double counting procedure for in-scattering contributions of group $g-1$ into g and g into g . The choice of ν_g for $\nu'_{min,g}$ guarantees that only photons with frequencies greater or equal than ν_g will be considered as in-scattering terms for group g . In-scattering contributions from group g into group g will be considered separately.

g	$h\nu_{g-1}$	$h\nu_g$	γ_g	$h\nu'_{max,g}$	$h\nu'_{min,g}$
1	-	5.000e+02	-	-	-
2	5.000e+02	6.400e+01	9.785e-01	8.539e+01	6.400e+01
3	6.400e+01	1.600e+01	1.252e-01	1.707e+01	1.600e+01
4	1.600e+01	4.000e+00	7.828e-03	4.064e+00	4.000e+00
5	4.000e+00	1.000e+00	1.957e-03	1.004e+00	1.000e+00
6	1.000e+00	2.500e-01	4.892e-04	2.502e-01	2.500e-01

TABLE III. Highest and smallest photon energies $h\nu'$ from which photons are able to scatter into group g . The photon energies are given in keV.

3. General case of the group transfer scattering cross section

First of all general solution for arbitrary photons energies by help of numerical methods are presented. Secondly, for small photon energies an analytical solution is derived.

The Compton differential cross section reads [1]

$$\Sigma_s(\nu' \rightarrow \nu, \mu_0) = \frac{3}{16\pi} \Sigma_{Th} \frac{(1 + \mu_0^2)}{(1 + \gamma'(1 - \mu_0))^2} \left(1 + \frac{\gamma'(1 - \mu_0)^2}{(1 + \mu_0^2)(1 + \gamma'(1 - \mu_0))} \right) \delta \left(\nu - \nu' \left(\frac{1}{1 + \gamma'(1 - \mu_0)} \right) \right).$$

Factors of the same order of μ_0 are grouped together. Hence, the Compton differential cross section reads

$$\begin{aligned} \Sigma_s(\nu' \rightarrow \nu, \mu_0) = & \frac{3}{16\pi} \Sigma_{Th} \frac{(1 + \gamma' + \gamma'^2) + (-\gamma' - 2\gamma'^2)\mu_0 + (1 + \gamma' + \gamma'^2)\mu_0^2}{(1 + \gamma'(1 - \mu_0))^3} \\ & - \frac{\gamma'\mu_0^3}{(1 + \gamma'(1 - \mu_0))^3} \times \delta \left(\nu - \frac{\nu'}{1 + \gamma'(1 - \mu_0)} \right). \end{aligned} \quad (37)$$

(37) is inserted into (28). In this way one obtains a triple integral. Since we are asking for the scattering probability of photons with all possible frequencies ν' for all possible angles μ_0 the integral over $d\nu$ is evaluated by means of the δ -distribution. The resulting fraction on the right hand side is abbreviated by

$$\frac{P(\mu_0, \gamma')}{R(\mu_0, \gamma')} = \frac{(1 + \gamma' + \gamma'^2) + (-\gamma' - 2\gamma'^2)\mu_0 + (1 + \gamma' + \gamma'^2)\mu_0^2 - \gamma'\mu_0^3}{(\gamma'^{-1} + 1 - \mu_0)^4}. \quad (38)$$

Hence, the moments of the Compton differential cross section read

$$S_{lg'g} = \frac{3}{8} \frac{\Sigma_{Th}}{B_{g'}} \frac{2(m_c^2)^4}{c^2 h^3} \Gamma(\gamma^{**}, \gamma^*) \quad (39)$$

where the results from the frequency shift formula (35) have been used. Here $\gamma^* = \gamma_g, \gamma^{**} = \gamma_{g+1}$ if $g' = g$ and $\gamma^* = \gamma_g/1 - 2\gamma_g, \gamma^{**} = \gamma_g$ if $g' = g - 1$. $\Gamma(\gamma^{**}, \gamma^*)$ is given by

$$\Gamma(\gamma^{**}, \gamma^*) = \int_{\gamma^{**}}^{\gamma^*} \frac{d\gamma'}{(\exp(\gamma' m_e c^2 / \theta) - 1) \gamma'} F_l(\gamma'), \quad (40)$$

$$F_l(\gamma') = \int_{-1}^1 d\mu_0 \frac{P(\mu_0, \gamma')}{R(\mu_0, \gamma')} P_l(\mu_0) = \sum_{i=1}^4 F_l^{(i)}(\gamma'), \quad (41)$$

where the integrals $F_l^{(i)}$ are defined by

$$F_l^{(1)}(\gamma') = \int_{-1}^1 d\mu_0 \frac{-\gamma' \mu_0^3}{(\gamma'^{-1} + 1 - \mu_0)^4} P_l(\mu_0) \quad (42)$$

$$F_l^{(2)}(\gamma') = \int_{-1}^1 d\mu_0 \frac{(1 + \gamma' + \gamma'^2) \mu_0^2}{(\gamma'^{-1} + 1 - \mu_0)^4} P_l(\mu_0) \quad (43)$$

$$F_l^{(3)}(\gamma') = \int_{-1}^1 d\mu_0 \frac{(-\gamma' - 2\gamma'^2) \mu_0}{(\gamma'^{-1} + 1 - \mu_0)^4} P_l(\mu_0) \quad (44)$$

$$F_l^{(4)}(\gamma') = \int_{-1}^1 d\mu_0 \frac{(1 + \gamma' + \gamma'^2)}{(\gamma'^{-1} + 1 - \mu_0)^4} P_l(\mu_0). \quad (45)$$

The integrals (42-45) are analytically evaluable. Taking the Legendre polynomials of zeroth order $F_0^{(i)}$ reads

$$\begin{aligned} F_0^{(1)}(\gamma') &= -\gamma' \left(\frac{6\tilde{a}}{1 - \tilde{a}^2} + \frac{6\tilde{a}^3}{(1 - \tilde{a}^2)^2} + \frac{2\tilde{a}^3 (3\tilde{a}^2 + 1)}{3 (1 - \tilde{a}^2)^3} + \ln \frac{|1 + \tilde{a}|}{|\tilde{a} - 1|} \right), \\ F_0^{(2)}(\gamma') &= -(1 + \gamma' + \gamma'^2) \left(\frac{2}{1 - \tilde{a}^2} + \frac{4\tilde{a}^2}{(1 - \tilde{a}^2)^2} + \frac{2\tilde{a}^2 (3\tilde{a}^2 + 1)}{3 (1 - \tilde{a}^2)^3} \right), \\ F_0^{(3)}(\gamma') &= -(\gamma' + 2\gamma'^2) \left(\frac{2\tilde{a}}{1 - \tilde{a}^2} + \frac{2\tilde{a} (3\tilde{a}^2 + 1)}{3 (1 - \tilde{a}^2)^3} \right), \\ F_0^{(4)}(\gamma') &= -\frac{2}{3} (1 + \gamma' + \gamma'^2) \frac{(3\tilde{a}^2 + 1)}{(1 - \tilde{a}^2)^3}, \end{aligned}$$

$\tilde{a} = -1/\gamma' - 1$. The integral functions $F_l^{(i)}$ of higher order in l are evaluated in the same manner. $F_l^{(i)}$ evaluated in this way is inserted in (39). The integration over γ' in $S_{lg'g}$ is carried out numerically. The results are given in tables (VII), (X) and (XIII) and are in agreement with the ones obtained by Pritzker et al. [4].

4. An asymptotic analytical solution

While $S_{lg'g}$ expressed by (28) is evaluable in a numerical way only, one can perform analytic calculations assuming the small photon energy limit $\gamma \ll 1$. In that case the differential scattering cross section $\Sigma_s(\nu' \rightarrow \nu, \mu_0)$ reads [1]

$$\begin{aligned} \Sigma_s(\nu' \rightarrow \nu, \mu_0) &= \frac{3}{16\pi} \Sigma_{Th} (1 + \mu_0^2) [1 - 2\gamma(1 - \mu_0) + \\ &\quad \gamma^2 \frac{(1 - \mu_0)^2 (4 + 3\mu_0^2)}{1 + \mu_0^2}] \times \delta(\nu' - \nu [1 - \gamma(1 - \mu_0) + \gamma^2(1 - \mu_0)^2]). \end{aligned} \quad (46)$$

This term is inserted into (28). The resulting triple integral is evaluated over $d\nu$ by means of the δ -distribution. In this way one obtains

$$S_{lg'g} = \frac{3}{8} \frac{\Sigma_{Th}}{B_{g'}} \frac{2(m_e c^2)^4}{c^2 h^3} \int_{-1}^1 d\mu_0 P_l(\mu_0) \int_{\gamma^{**}}^{\gamma^*} \frac{d\gamma' \gamma'^3}{(\exp(\gamma' m_e c^2 / \theta) - 1)} \times \{ (1 - 3\gamma' + 7\gamma'^2 - 6\gamma'^3 + 4\gamma'^4) + \\ + (3\gamma' - 14\gamma'^2 + 18\gamma'^3 - 16\gamma'^4) \mu_0 + (1 - 3\gamma' + 13\gamma'^2 - 23\gamma'^3 + 27\gamma'^4) \mu_0^2 + \\ + (3\gamma' - 12\gamma'^2 + 21\gamma'^3 - 28\gamma'^4) \mu_0^3 + (6\gamma'^2 - 15\gamma'^3 + 22\gamma'^4) \mu_0^4 + (5\gamma'^3 - 12\gamma'^4) \mu_0^5 + 3\gamma'^4 \mu_0^6 \}. \quad (47)$$

In lowest Legendre order, $P_0(\mu_0) = 1$, the above expression integrated over μ_0 leads to

$$S_{0g'g} = \frac{3}{8} \frac{\Sigma_{Th}}{B_{g'}} \frac{2\theta^4}{c^2 h^3} \left\{ \frac{8}{3} \mathcal{I}_3(u^{**}, u^*) - 8\alpha \mathcal{I}_4(u^{**}, u^*) + \frac{376}{15} \alpha^2 \mathcal{I}_5(u^{**}, u^*) \right. \\ \left. - \frac{100}{3} \alpha^3 \mathcal{I}_6(u^{**}, u^*) + \frac{1248}{35} \alpha^4 \mathcal{I}_7(u^{**}, u^*) \right\} \quad (48)$$

where $\alpha = \theta / m_e c^2$ and $u^* = \gamma^* / \alpha$, $u^{**} = \gamma^{**} / \alpha$. The terms \mathcal{I}_k are derived in the appendix. The result is presented in table (VIII). Proceeding with the next order group moment coefficient, $P_1(\mu_0) = \mu_0$, yields

$$S_{1g'g} = \frac{3}{8} \frac{\Sigma_{Th}}{B_{g'}} \frac{2\theta^4}{c^2 h^3} \left\{ \frac{16}{5} \alpha \mathcal{I}_4(u^{**}, u^*) - \frac{212}{15} \alpha^2 \mathcal{I}_5(u^{**}, u^*) + \frac{764}{35} \alpha^3 \mathcal{I}_6(u^{**}, u^*) - \frac{236}{5} \alpha^4 \mathcal{I}_7(u^{**}, u^*) \right\}. \quad (49)$$

The result is presented in table (XI). Finally, the second order group moment, $P_2(\mu_0) = 0.5(-1 + 3\mu_0^2)$, is considered.

$$S_{2g'g} = -\frac{1}{2} S_{0g'g} + \frac{9}{16} \frac{\Sigma_{Th}}{B_{g'}} \frac{2\theta^4}{c^2 h^3} \left\{ \frac{16}{15} \mathcal{I}_3(u^{**}, u^*) - \frac{16}{5} \alpha \mathcal{I}_4(u^{**}, u^*) \right. \\ \left. + \frac{1216}{105} \alpha^2 \mathcal{I}_5(u^{**}, u^*) - \frac{612}{35} \alpha^3 \mathcal{I}_6(u^{**}, u^*) + \frac{2018}{105} \alpha^4 \mathcal{I}_7(u^{**}, u^*) \right\}. \quad (50)$$

The result is presented in table (XIV). One can see by the tables (IX), (XII) and (XV) the deviation from the numerically evaluated results is very small except for the highest energy group. Comparing to the published results from Pritzker one obtains the same results without much numerical effort.

III. CONCLUSION AND DISCUSSION

In this paper, numerical and analytical expressions for the multi-group total scattering and transfer scattering cross section are found. Scattering is analysed by Thomson and Compton scattering for free electrons at rest. Analytical terms have been obtained in the small photon energy limit. From tables (IX), (XII) and (XV) one can see that the deviation from the general valid results, shown in tables (VII), (X) and (XIII), is a few percents only. The Compton cross section starts to deviate from the Thomson limit at high temperatures and high energy groups, e.g. tables (IV) and (VII). For temperatures below 3 keV and energy groups with appreciable contributions to the Planck spectrum these deviations are small. The moment $l = 1$, which vanishes in the Thomson limit, should be taken into account as like as the transfer cross sections S_{lg-1g} which become comparable to S_{lgg} for temperatures above 1 keV. As pointed out by Pritzker et al. [6] the diffusion technique is an adequate approximation describing the radiative processes in a plasma at temperatures below 3 keV. In that case the plasma is optical thick and scattering is not important. For higher temperatures the plasma becomes optical thin. In such regions Compton scattering should be considered for a better approximation of the ERT.

IV. ACKNOWLEDGEMENT

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APPENDIX

The solution of integrals of the type

$$\mathcal{I}_k(a, b) = \frac{1}{k!} \int_a^b \frac{du u^k}{\exp(u) - 1} \quad (51)$$

taking $a = 0$ and $b = \infty$ is given by the Zeta-function

$$\zeta(k) = \sum_{n=1}^{\infty} \frac{1}{n^{k+1}}. \quad (52)$$

Such integrals appear in one-energy group calculations. More effort is required setting a and b to arbitrary real values. A lengthy technical integration procedure yields to

$$\mathcal{I}_k(a, b) = \sum_{n=1}^{\infty} \frac{1}{n^{k+1}} \left[\exp(-\tilde{a}) \left(\sum_{i=0}^k \frac{k!}{i!} \tilde{a}^i \right) - \exp(-\tilde{b}) \left(\sum_{i=0}^k \frac{k!}{i!} \tilde{b}^i \right) \right], \quad (53)$$

where $\tilde{a} = na$ and $\tilde{b} = nb$.

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g/θ	1.00e-01	3.16e-01	1.00e+00	3.16e+00	1.00e+01
1	8.140e-01	8.136e-01	8.121e-01	8.067e-01	7.851e-01
2	9.420e-01	9.413e-01	9.385e-01	9.260e-01	8.831e-01
3	9.843e-01	9.832e-01	9.781e-01	9.635e-01	9.574e-01
4	9.956e-01	9.937e-01	9.899e-01	9.887e-01	9.884e-01
5	9.982e-01	9.973e-01	9.971e-01	9.971e-01	9.970e-01
6	9.996e-01	9.994e-01	9.994e-01	9.994e-01	9.994e-01

TABLE IV. Six-group Planck mean Compton cross section in Thomson units evaluated at different temperatures θ in keV.

g/θ	1.00e-01	3.16e-01	1.00e+00	3.16e+00	1.00e+01
1	8.309e-01	8.306e-01	8.297e-01	8.266e-01	8.191e-01
2	9.421e-01	9.414e-01	9.386e-01	9.265e-01	8.871e-01
3	9.843e-01	9.832e-01	9.780e-01	9.635e-01	9.574e-01
4	9.956e-01	9.937e-01	9.899e-01	9.887e-01	9.884e-01
5	9.982e-01	9.973e-01	9.971e-01	9.971e-01	9.970e-01
6	9.996e-01	9.994e-01	9.994e-01	9.994e-01	9.994e-01

TABLE V. The table shows the six-group Planck mean Compton cross section in Thomson units evaluated at different temperatures θ in keV. The constants Σ_{stg}/Σ_{Th} obtained from (19) are independent from material and density. The results are in good agreement with those in table (IV) except for the case, where $\gamma \ll 1$ is not valid.

g/θ	1.00e-01	3.16e-01	1.00e+00	3.16e+00	1.00e+01
1	2.077	2.095	2.172	2.474	4.322
2	0.010	0.011	0.016	0.053	0.456
3	0.000	0.000	0.001	0.000	0.001
4	0.000	0.000	0.000	0.000	0.000
5	0.000	0.000	0.000	0.000	0.000
6	0.000	0.000	0.000	0.000	0.000

TABLE VI. Accuracy of the multi-group Compton cross section in small photon energy limit. The table shows the deviation in percent from the general valid multi-group term (16). Precise agreement between the general case (table (V)) and small energy limit (table (IV)) are obtained for the energy groups two to five up to temperatures of around 4 keV.

$g', g/\theta$	1.00e-01	3.16e-01	1.00e+00	3.16e+00	1.00e+01
0,1	-	-	-	-	-
1,1	7.309e-01	7.302e-01	7.281e-01	7.206e-01	6.910e-01
1,2	7.309e-01	7.302e-01	7.281e-01	7.189e-01	5.329e-01
2,2	9.140e-01	9.129e-01	9.087e-01	8.906e-01	8.287e-01
2,3	9.140e-01	8.756e-01	5.366e-01	1.540e-01	2.059e-02
3,3	9.765e-01	9.749e-01	9.672e-01	9.457e-01	9.366e-01
3,4	4.354e-01	1.427e-01	2.812e-02	3.481e-03	1.176e-03
4,4	9.934e-01	9.906e-01	9.849e-01	9.831e-01	9.827e-01
4,5	2.821e-02	4.664e-03	6.242e-04	2.647e-04	2.073e-04
5,5	9.973e-01	9.960e-01	9.957e-01	9.956e-01	9.956e-01
5,6	7.448e-04	1.194e-04	6.118e-05	5.062e-05	4.783e-05
6,6	9.992e-01	9.991e-01	9.990e-01	9.990e-01	9.990e-01

TABLE VII. Zeroth order moments of the six-group Planck weighted Compton scattering transfer cross section. The results are given in units of the Thomson cross section $S_{lg'g}/\Sigma_{Th}$. The temperature θ is given in units of keV.

$g', g/\theta$	1.00e-01	3.16e-01	1.00e+00	3.16e+00	1.00e+01
0,1	-	-	-	-	-
1,1	7.502e-01	7.497e-01	7.482e-01	7.430e-01	7.264e-01
1,2	7.502e-01	7.497e-01	7.482e-01	7.411e-01	5.533e-01
2,2	9.144e-01	9.133e-01	9.093e-01	8.916e-01	8.341e-01
2,3	9.144e-01	8.760e-01	5.369e-01	1.541e-01	2.060e-02
3,3	9.765e-01	9.749e-01	9.673e-01	9.459e-01	9.368e-01
3,4	4.354e-01	1.427e-01	2.812e-02	3.481e-03	1.176e-03
4,4	9.934e-01	9.906e-01	9.849e-01	9.831e-01	9.827e-01
4,5	2.821e-02	4.664e-03	6.242e-04	2.647e-04	2.073e-04
5,5	9.973e-01	9.960e-01	9.957e-01	9.956e-01	9.956e-01
5,6	7.448e-04	1.194e-04	6.118e-05	5.062e-05	4.783e-05
6,6	9.992e-01	9.991e-01	9.990e-01	9.990e-01	9.990e-01

TABLE VIII. Zeroth order moment of the six-group Planck weighted Compton scattering transfer cross section in the case of small photon energies. The results are given in units of the Thomson cross section $S_{lg'g}/\Sigma_{Th}$. The approximation fails at high photon energy groups and high temperatures. In all other cases the results are in good agreement with those obtained by numerical integration. Refer to table (VII) additionally.

$g', g/\theta$	1.00e-01	3.16e-01	1.00e+00	3.16e+00	1.00e+01
0,1	-	-	-	-	-
1,1	2.651	2.673	2.761	3.106	5.120
1,2	2.646	2.673	2.761	3.097	3.813
2,2	0.047	0.049	0.057	0.119	0.652
2,3	0.047	0.048	0.050	0.050	0.051
3,3	0.001	0.001	0.003	0.014	0.021
3,4	0.001	0.001	0.001	0.001	0.001
4,4	0.000	0.000	0.000	0.000	0.000
4,5	0.000	0.000	0.000	0.000	0.000
5,5	0.000	0.000	0.000	0.000	0.000
5,6	0.000	0.000	0.000	0.000	0.000
6,6	0.000	0.000	0.000	0.000	0.000

TABLE IX. Accuracy of the multi-group Compton transfer cross section to zeroth order in the small photon energy limit. The deviation is given in percent comparing the general valid term (39) with small photon energy approximation (48). The results are in agreement to those achieved by the general valid expression (39) except for the highest energy groups.

$g', g/\theta$	1.00e-01	3.16e-01	1.00e+00	3.16e+00	1.00e+01
0,1	-	-	-	-	-
1,1	9.291e-02	9.310e-02	9.370e-02	9.573e-02	1.032e-01
1,2	9.291e-02	9.310e-02	9.370e-02	9.544e-02	7.486e-02
2,2	3.307e-02	3.348e-02	3.499e-02	4.148e-02	6.237e-02
2,3	3.307e-02	3.203e-02	1.982e-02	5.711e-03	7.642e-04
3,3	9.313e-03	9.938e-03	1.290e-02	2.114e-02	2.459e-02
3,4	4.074e-03	1.336e-03	2.633e-04	3.260e-05	1.101e-05
4,4	2.632e-03	3.742e-03	5.995e-03	6.712e-03	6.873e-03
4,5	6.619e-05	1.094e-05	1.465e-06	6.212e-07	4.865e-07
5,5	1.072e-03	1.591e-03	1.726e-03	1.757e-03	1.766e-03
5,6	4.372e-07	7.010e-08	3.591e-08	2.971e-08	2.807e-08
6,6	4.010e-04	4.298e-04	4.368e-04	4.387e-04	4.393e-04

TABLE X. First order moments of the six-group Planck weighted Compton scattering transfer cross section. The results are given in units of the Thomson cross section $S_{lg'g}/\Sigma_{Th}$. The temperature θ is given in units of keV.

$g', g/\theta$	1.00e-01	3.16e-01	1.00e+00	3.16e+00	1.00e+01
0,1	-	-	-	-	-
1,1	7.891e-02	7.895e-02	7.909e-02	7.933e-02	7.566e-02
1,2	7.891e-02	7.895e-02	7.909e-02	7.914e-02	5.972e-02
2,2	3.279e-02	3.319e-02	3.465e-02	4.078e-02	5.861e-02
2,3	3.279e-02	3.175e-02	1.965e-02	5.660e-03	7.574e-04
3,3	9.308e-03	9.932e-03	1.289e-02	2.105e-02	2.446e-02
3,4	4.072e-03	1.335e-03	2.631e-04	3.258e-05	1.101e-05
4,4	2.632e-03	3.742e-03	5.993e-03	6.710e-03	6.871e-03
4,5	6.618e-05	1.094e-05	1.465e-06	6.212e-07	4.865e-07
5,5	1.072e-03	1.591e-03	1.726e-03	1.757e-03	1.766e-03
5,6	4.372e-07	7.010e-08	3.591e-08	2.971e-08	2.813e-08
6,6	4.010e-04	4.298e-04	4.368e-04	4.387e-04	4.393e-04

TABLE XI. First order moments of the six-group Planck weighted Compton scattering transfer cross section for the case of small photon energies. The results are given in units of the Thomson cross section $S_{lg'g}/\Sigma_{Th}$. The approximation fails at high photon energy groups and high temperatures. In all other cases the results are in very good agreement with those obtained by numerical integration. Refer to table (X) additionally.

$g', g/\theta$	1.00e-01	3.16e-01	1.00e+00	3.16e+00	1.00e+01
0,1	-	-	-	-	-
1,1	15.072	15.192	15.584	17.129	26.703
1,2	15.076	15.192	15.584	17.084	20.218
2,2	0.846	0.872	0.980	1.702	6.019
2,3	0.846	0.867	0.885	0.892	0.894
3,3	0.054	0.063	0.134	0.412	0.521
3,4	0.052	0.052	0.052	0.052	0.052
4,4	0.004	0.011	0.028	0.033	0.034
4,5	0.003	0.003	0.003	0.003	0.003
5,5	0.001	0.002	0.002	0.002	0.002
5,6	0.000	0.000	0.000	0.000	0.207
6,6	0.000	0.000	0.000	0.000	0.000

TABLE XII. Accuracy of the multi-group Compton transfer cross section to first order in the small photon energy limit. The deviation in percent comparing the general valid term (39) with the small photon energy approximation (49). The results are in very good agreement to those achieved by the general valid expression (39) except for the highest energy groups.

$g', g/\theta$	1.00e-01	3.16e-01	1.00e+00	3.16e+00	1.00e+01
0,1	-	-	-	-	-
1,1	8.041e-02	8.039e-02	8.029e-02	7.997e-02	7.890e-02
1,2	8.041e-02	8.039e-02	8.029e-02	7.977e-02	5.987e-02
2,2	9.213e-02	9.204e-02	9.170e-02	9.029e-02	8.605e-02
2,3	9.213e-02	8.827e-02	5.411e-02	1.553e-02	2.076e-03
3,3	9.770e-02	9.755e-02	9.684e-02	9.489e-02	9.408e-02
3,4	4.356e-02	1.428e-02	2.814e-03	3.483e-04	1.177e-04
4,4	9.934e-02	9.907e-02	9.851e-02	9.834e-02	9.830e-02
4,5	2.821e-03	4.664e-04	6.242e-05	2.647e-05	2.073e-05
5,5	9.973e-02	9.960e-02	9.957e-02	9.956e-02	9.956e-02
5,6	7.448e-05	1.194e-05	6.118e-06	5.062e-06	4.783e-06
6,6	9.992e-02	9.991e-02	9.990e-02	9.990e-02	9.990e-02

TABLE XIII. Second order moments of the six-group Planck weighted Compton scattering transfer cross section. The results are given in units of the Thomson cross section $S_{lg'g}/\Sigma_{Th}$. The temperature θ is given in units of keV.

$g', g/\theta$	1.00e-01	3.16e-01	1.00e+00	3.16e+00	1.00e+01
0,1	-	-	-	-	-
1,1	8.486e-02	8.487e-02	8.490e-02	8.507e-02	8.665e-02
1,2	8.486e-02	8.487e-02	8.490e-02	8.484e-02	6.443e-02
2,2	9.224e-02	9.215e-02	9.184e-02	9.056e-02	8.735e-02
2,3	9.224e-02	8.838e-02	5.418e-02	1.555e-02	2.079e-03
3,3	9.770e-02	9.755e-02	9.684e-02	9.492e-02	9.413e-02
3,4	4.357e-02	1.428e-02	2.814e-03	3.483e-04	1.177e-04
4,4	9.934e-02	9.907e-02	9.852e-02	9.834e-02	9.830e-02
4,5	2.821e-03	4.664e-04	6.242e-05	2.647e-05	2.073e-05
5,5	9.973e-02	9.960e-02	9.957e-02	9.956e-02	9.956e-02
5,6	7.448e-05	1.194e-05	6.118e-06	5.062e-06	4.783e-06
6,6	9.992e-02	9.991e-02	9.990e-02	9.990e-02	9.990e-02

TABLE XIV. Second order moments of the six-group Planck weighted Compton scattering transfer cross section in the case of small photon energies. The results are given in units of the Thomson cross section $S_{lg'g}/\Sigma_{Th}$. The approximation fails at high photon energy groups and high temperatures. In all other cases the results are in very good agreement with those obtained by numerical integration. Refer to table (XIII) additionally.

$g', g/\theta$	1.00e-01	3.16e-01	1.00e+00	3.16e+00	1.00e+01
0,1	-	-	-	-	-
1,1	5.533	5.577	5.742	6.371	9.824
1,2	5.528	5.577	5.742	6.355	7.631
2,2	0.120	0.125	0.147	0.298	1.505
2,3	0.120	0.124	0.128	0.129	0.130
3,3	0.002	0.003	0.007	0.037	0.054
3,4	0.002	0.002	0.002	0.002	0.002
4,4	0.000	0.000	0.001	0.001	0.001
4,5	0.000	0.000	0.000	0.000	0.000
5,5	0.000	0.000	0.000	0.000	0.000
5,6	0.000	0.000	0.000	0.000	0.000
6,6	0.000	0.000	0.000	0.000	0.000

TABLE XV. Accuracy of the multi-group Compton transfer cross section of second order in small photon energy limit. The deviation is given in percent comparing the general valid term (39) with the small photon energy approximation (50). The results are in very good agreement with those achieved by the general valid expression (39) except for the highest energy groups.